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SPATIALLY PERIODIC FUNDAMENTAL SOLUTIONS OF THE THEORY OF OSCILLATIONS†

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Spatially periodic fundamental solutions of the theory of oscillations are constructed, applicable to anisotropic elastic media with a general form of anisotropy. The results are compared with the isotropic case.

Spatially periodic fundamental solutions for the Helmholtz equation were apparently first constructed by Titchmarsh by expansion in Fourier series [1]. This method was employed in [2] in the case of spatially periodic equations of the statics of an anisotropic body. Fundamental solutions for (spatially) periodic problems of the theory of oscillations in the case of an isotropic medium were obtained in [3].

1. FUNDAMENTAL OPERATORS

We will consider an elastically anisotropic uniform medium in R^3, the equations of the oscillations of which have the form

$$\mathbf{A}(\partial_x, \omega)\,\mathbf{u} \equiv -\operatorname{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u} - \omega^2 \mathbf{u} = 0 \tag{1.1}$$

where $\mathbf{A}$ is a matrix differential operator of the theory of oscillations, $\mathbf{C}$ is the four-valent elasticity tensor, $\mathbf{u}$ is the displacement field in the medium and ω is the frequency of the oscillations.

We will assume that the tensor $\mathbf{C}$ is strictly elliptic, which ensures that the operator $\mathbf{A}(\partial_x, 0)$ is strictly elliptic. The medium is assumed to be hyperelastic, which guarantees that the tensor $\mathbf{C}$ is symmetrical if it is regarded as an operator which acts in the space of symmetric second-rank tensors.

The Fourier integral transformation

$$f^\wedge(\xi) = \int_{R^3} f(x)\exp(2\pi i\xi\cdot\mathbf{x})\,dx$$

applied to the operator $\mathbf{A}$ enables us to obtain the corresponding symbol

$$\mathbf{A}^\wedge(\xi, \omega) = (2\pi)^2 \xi\cdot\mathbf{C}\cdot\xi - \omega^2 \mathbf{I} \tag{1.2}$$

where $\mathbf{I}$ is the unit diagonal matrix.

From (1.2) one can obtain the symbol of a spatially non-periodic fundamental solution of the equation of the oscillation theory (the Fourier transform of the fundamental solutions)

$$\mathbf{E}^\wedge(\xi, \omega) = \mathbf{A}^\wedge(\xi, \omega)^{-1} \tag{1.3}$$

In the general case of anisotropy, the Fourier transformation of expression (1.3) can only be carried out numerically [2], but for spatially periodic fundamental solutions it is sufficient to determine the symbol $\mathbf{E}^\wedge$.

2. CONSTRUCTION OF A PERIODIC FUNDAMENTAL SOLUTION

Consider the identity

$$\mathbf{A}(\partial_x, \omega)\,\mathbf{E}_p(\mathbf{x}, \omega) = \delta_p(\mathbf{x})\,\mathbf{I} \tag{2.1}$$

where the subscript p indicates spatial periodicity. By analogy with the approach employed previously in [2], we will represent the periodic δ-function in the form of a series

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$$\delta_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda^*} \exp(2\pi i \mathbf{m}^* \cdot \mathbf{x}) \tag{2.2}$$

where V_Q is the volume of the fundamental region, Λ^* is the grid of the conjugate basis, and $\mathbf{m}^*$ are the conjugate nodes.

Expanding the fundamental solution $\mathbf{E}_p$ in the form of a similar series, we obtain from (2.1)

$$\mathbf{E}_p(\mathbf{x},\omega) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda^*} \mathbf{E}^\wedge(\mathbf{m}^*,\omega)\exp(2\pi i \mathbf{m}^* \cdot \mathbf{x}) \tag{2.3}$$

Hence, to construct a spatially periodic fundamental solution it is sufficient to know the symbol $\mathbf{E}_p$, which is easily calculated from (1.3).

Note. For a simple cubic structural grid with vectors of the fundamental basis of unit length, the vectors of the conjugate basis form a conjugate structural grid, which coincides with the initial grid. In this case it is natural to carry out the summation in (2.2) and (2.3) over the nodal points of the grid of the fundamental basis.

Assertions 1. The series on the right-hand side of (2.3) is absolutely divergent for any values of ω and $\mathbf{x}$.

The proof of Assertion 1 follows from the general scheme of the proof of the formula for Poisson summation for periodic functions [4]. The proof of Assertion 2 follows from the divergence in R^3 of majorant multiple series with the asymptotic estimate

$$|f_m| = O(|m|^{-2}), \quad |m| \to \infty, \quad \mathbf{m} \in \Lambda \tag{2.4}$$

where Λ is an arbitrary structural grid in R^3. The estimate (2.4) in turn follows directly from (1.2) and (1.3).

3. PERIODIC FUNDAMENTAL SOLUTION OF THE OSCILLATION THEORY FOR AN ISOTROPIC ELASTIC MEDIUM

The equations of the oscillation theory for the operator of the theory of elasticity in the isotropic case in tensor notation have the form

$$(\lambda + \mu)\, u_{j,ij} + \mu\, \Delta u_i + \rho\omega^2 u_i = 0, \quad i, j = 1, 2, 3 \tag{3.1}$$

where λ and μ are the Lamé coefficients and ρ is the density of the medium, and summation is carried out over repeated subscripts from 1 to 3. The components of the fundamental solution E_{mi} satisfy the following identity

$$(\lambda + \mu)\, E_{mj,ij} + \mu\, \Delta E_{mi} + \rho\omega^2 E_{mi} = \delta(\mathbf{x})\, \delta_{mi} \tag{3.2}$$

Substituting into (3.2) the expansion of $\delta(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ in Fourier series

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathbf{k}}^{\infty} e^{i(\mathbf{kx})}, \quad E_{mi}(\mathbf{x}) = \sum_{\mathbf{k}}^{\infty} E_{mi}^{\mathbf{k}} e^{i(\mathbf{kx})}, \quad \mathbf{k} = (k_1, k_2, k_3) \tag{3.3}$$

we obtain that the components $E_{mi}^{\mathbf{k}}$ must satisfy the following system of linear equations

$$E_{jm}^{\mathbf{k}} A_{ij}^{\wedge} = \delta_{mi} \tag{3.4}$$

$$A_{ij}^{\wedge} = \mu(2\pi)^3(\delta_{ij}\beta^2 - \delta_{ij}|k|^2 - (\eta^{-2} - 1)k_i k_j)$$

$$\beta = \omega / C_2, \quad C_1 = [(\lambda + 2\mu)/\rho]^{1/2}, \quad C_2 = [\mu/\rho]^{1/2}, \quad \eta = C_2/C_1$$

where C_1 and C_2 are the velocities of propagation of longitudinal and transverse waves in the elastic medium. In (3.3) we have assumed that the parameter $\mathbf{k}$ belongs to the grid of the conjugate basis, formed by vectors of length 2π.

Solving the system of linear equations (3.4) for $A_{ij}^{\wedge}$, we obtain

$$E_{11}^{\mathbf{k}} = \frac{1}{\eta^2 D}(\beta^2\eta^2 - \eta^2 k_1^2 - k_2^2 + k_2 k_3(1 - \eta^2))(\beta^2\eta^2 - \eta^2 k_1^2 - k_2^2 - k_2 k_3(1 - \eta^2))$$

$$E_{22}^{\mathbf{k}} = \frac{1}{\eta^2 D}(\beta^2\eta^2(\beta^2\eta^2 - k_1^2 - k_2^2 - \eta^2) + \eta^2(k_1^4 + k_2^4) + k_1^2 k_2^2(1 + \eta^4) + k_1^4 k_3^4(1 - \eta^2)^2)$$

$$E_{33}^{\mathbf{k}} = \frac{1}{D}(\beta^2 - k_1^2 - k_2^2)(\beta^2\eta^2 - k_1^2 - k_2^2) \tag{3.5}$$

$$E_{12}^{\mathbf{k}} = \frac{1}{\eta^2 D}(\eta^2 - 1)k_1 k_2(-\beta^2\eta^2 + \eta^2 k_1^2 + k_2^2 - k_3^2(1-\eta^2))$$

$$E_{n3}^{\mathbf{k}} = \frac{1}{D}(\eta^2 - 1)k_n k_3(-\beta^2 + k_1^2 + k_2^2), \quad n = 1,2$$

$$E_{ij}^{\mathbf{k}} = E_{ji}^{\mathbf{k}}, \quad i \neq j$$

$$D = \mu(2\pi)^3(\beta^2 - k_1^2 - k_2^2)((\beta^2\eta^2 - k_2^2)^2 - \eta^2 k_1^2(\beta^2 + \beta^2\eta^2 - k_1^2) -$$

$$-k_1^2 k_2^2(1+\eta^2) - (1-\eta^2)^2 k_3^2(k_1^2 + k_2^2))$$

The components of the spatially periodic fundamental solution of the oscillation theory for an isotropic medium are given by (3.5).

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